Corrigé 3

Exercice 1. Trouver les fonctions analytiques $f(z) = u(x, y) + iv(x, y)$ qui ont pour partie réelle ou imaginaire :

(1) $u(x, y) = x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}$;

(2) $u(x, y) = e^x(x \cos y - y \sin y) + 2 \sin x \sinh y + x^3 - 3xy^2 + y$;

(3) $v(x, y) = 3 + x^2 - y^2 - \frac{y}{2(x^2+y^2)}$;

(4) $v(x, y) = \log(x^2 + y^2) + x - 2y$.

Démonstration. We will write the real and imaginary parts as a linear combinations of real or imaginary parts of simple analytic functions. To achieve the desired result we try to match $u(x, y)$ and $v(x, y)$ by polynomials or trigonometric functions in z that have a similar form and show that the desired function is the real or imaginary part of the given linear combination. This proves the existence of $f(z)$. The Cauchy-Riemann relations then imply that this function is unique up to a constant.

(1) We start with the term $x^2 - y^2$. Since this is a quadratic polynomial in x, y it is natural to consider the real and imaginary parts of z^2, where $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$. We have $\mathrm{Re}(z^2) = x^2 - y^2$, hence we try to find the right linear combination of real parts. The next term is $5x + y$, a degree 1 polynomial, and a natural candidate is z. We have $\mathrm{Re}(5z) = 5x$ and in order to get the y term we 'swap' the real and imaginary parts by multiplying with i, which gives $\mathrm{Re}(iz) = -y$ and finally $\mathrm{Re}(5z - iz) = 5x + y$. For the last term, we note that $x^2 + y^2 = |z|^2 = z\bar{z}$ and $-y = \mathrm{Re}(-i\bar{z})$, this gives us $\mathrm{Re}(\frac{-i}{z}) = \mathrm{Re}(\frac{-i\bar{z}}{z\bar{z}}) = \frac{-y}{x^2+y^2}$. Using the linearity of the real part, we conclude that $f(z) = z^2 + 5z - iz - \frac{i}{z} + iC$ has the desired property, where $C \in \mathbb{R}$ is an arbitrary constant.

(2) We start with the degree 3 polynomial $x^3 - 3xy^2$ for which a natural guess is z^3. A simple calculation gives $\mathrm{Re}(z^3) = x^3 - 3xy^2$. For the linear part our guess is again z . For any complex function h we have $\mathrm{Re}(ih(z)) = \mathrm{Re}(i(u(x, y) + iv(x, y))) = -v(x, y)$ which implies $\mathrm{Re}(-iz) = y$. For the trigonometric part, a natural thing to do is to look at $\cos(z)$. We have $2 \cos(z) = e^{iz} + e^{-iz} = e^{-y}e^{ix} + e^{y}e^{-ix} = -2i \sinh(y) \sin(x) + 2 \cosh(y) \cos(x)$. Using the previous trick and multiplying by i, we get $\mathrm{Re}(2i \cos(z)) = 2 \sinh(y) \sin(x)$. The last term is harder to guess. A natural guess would be to consider e^z times a simple function. It turns out that ze^z is the correct function and we have $\mathrm{Re}(ze^z) = e^x(x \cos y - y \sin y)$. Finally we conclude that $f(z) = z^3 - iz + 2i \cos(z) + ze^z + iC$.

(3) From the first exercise we know that $\mathrm{Re}(z^2) = x^2 - y^2$ and $\frac{1}{2}\mathrm{Re}(\frac{-i}{z}) = \frac{-y}{2(x^2+y^2)}$. Since we are asked to find the imaginary part of $f(z)$, we use the fact that $\mathrm{Im}(ih(z)) = \mathrm{Im}(i(u(x, y) + iv(x, y))) = u(x, y)$ to swap the real and imaginary part. We conclude that $\mathrm{Im}(iz^2) = x^2 - y^2$ and $\frac{1}{2}\mathrm{Im}(\frac{-ii}{z}) = \frac{-y}{2(x^2+y^2)}$. Finally, we have $f(z) = iz^2 + \frac{1}{2z} + 3i + C$.

(4) Similar to before, the $x - 2y$ term can be written by using $-2\mathrm{Im}(z) = -2y$ and using the 'swap' from before by multiplying with i, i.e., $\mathrm{Im}(iz) = x$, hence $x - 2y = \mathrm{Im}(-2z + iz)$. For the term $\log(x^2 + y^2)$, recall that $\mathrm{Re}\Big(\log(z)\Big) = \log(|z|) = \log(\sqrt{x^2 + y^2})$, which together with the 'swap' gives us $\mathrm{Im}\Big(2i \log(z)\Big) = 2 \log(|z|) = \log(x^2 + y^2)$. We conclude that $f(z) = -2z + iz + 2i \log(z) + C$.

□

Exercice 2. Trouver une fonction continue $f : \mathbb{C} \to \mathbb{C}$ holomorphe en seulement un point.

Démonstration. Les parties réelle et imaginaire de la fonction $f(z) = |z|^2 = x^2+y^2$ ont comme dérivées partielles

$$\partial_x u(x, y) = 2x, \qquad \partial_y u(x, y) = 2y, \qquad \partial_x v(x, y) = \partial_y v(x, y) = 0.$$

Les équations de Cauchy-Riemann sont donc satisfaites seulement quand $x = y = 0$. □

Exercice 3. Déterminer où les fonctions suivantes sont holomorphes :

(1) $f(x + iy) = x^2 + y^2 + 2ixy$;

(2) $f(z) = z\mathrm{Re}z$;

(3) $f(z) = e^z$;

(4) $f(z) = \bar{z}$;

Démonstration. (1) On a $\partial_x u(x,y) = 2x$, $\partial_y u(x,y) = 2y$, $\partial_x v(x,y) = 2y$ et $\partial_y v(x,y) = 2x$. Les équations de Cauchy-Riemann sont satisfaites pour $y = 0$ et tout $x \in \mathbb{R}$.

(2) On a $\partial_x u(x,y) = 2x$, $\partial_y u(x,y) = 0$, $\partial_x v(x,y) = y$ et $\partial_y v(x,y) = x$. Les équations de Cauchy-Riemann sont satisfaites pour $x = 0$ et $y = 0$.

(3) On a $\partial_x u(x,y) = e^x \cos y$, $\partial_y u(x,y) = -e^x \sin y$, $\partial_x v(x,y) = e^x \sin y$ et $\partial_y v(x,y) = e^x \cos y$. Les équations de Cauchy-Riemann sont satisfaites pour tout $x, y \in \mathbb{R}$.

(4) On a $\partial_x u(x,y) = 1$, $\partial_y u(x,y) = 0$, $\partial_x v(x,y) = 0$ et $\partial_y v(x,y) = -1$. Les équations de Cauchy-Riemann ne sont donc satisfaites pour aucun $x, y \in \mathbb{R}$.

□

<u>Exercice 4.</u> Soit $f : U \to \mathbb{C}$ une fonction $\mathcal{C}^1$ (au sens de fonctions de $\mathbb{R}^2$ vers $\mathbb{R}^2$). Montrer que f est holomorphe sur U si et seulement si $\bar{\partial} f(z) = 0$ pour tout $z \in U$ et que dans ce cas $f'(z) = \partial f(z)$.

Démonstration. On a

$$\begin{aligned}
\overline{\partial} f(z) &= \frac{1}{2}\left(\partial_x f(z) + i\partial_y f(z)\right) \\
&= \frac{1}{2}\left(\partial_x u(x,y) + i\partial_y u(x,y)\right) + \frac{i}{2}\left(\partial_x v(x,y) + i\partial_y v(x,y)\right) \\
&= \frac{1}{2}\left(\partial_x u(x,y) - \partial_y v(x,y)\right) + \frac{i}{2}\left(\partial_y u(x,y) + \partial_x v(x,y)\right).
\end{aligned}$$

On voit que les parties réelle et imaginaire sont 0 si et seulement si les équations de Cauchy-Riemann sont satisfaites. On a alors,

$$\begin{aligned}
\partial f(z) &= \frac{1}{2}\left(\partial_x f(z) - i\partial_y f(z)\right) \\
&= \frac{1}{2}\left(\partial_x u(x,y) + i\partial_x v(x,y)\right) - \frac{i}{2}\left(\partial_y u(x,y) + i\partial_y v(x,y)\right) \\
&= \frac{1}{2}\left(\partial_x u(x,y) + \partial_y v(x,y)\right) + \frac{i}{2}\left(-\partial_y u(x,y) + \partial_x v(x,y)\right) \\
&= \partial_x u(x,y) + i\partial_x v(x,y) \\
&= \partial_x f(x+iy) = f'(z).
\end{aligned}$$

□

<u>Exercice 5.</u> Montrer que si $f : U \to V$ est bijective et holomorphe et si f' ne s'annule pas sur V, alors la fonction inverse f^{-1} est aussi holomorphe.

Démonstration. En considérant la fonction f comme allant de $\mathbb{R}^2$ à $\mathbb{R}^2$, sa Jacobienne $Df\big|_x$ est donc une matrice D de taille 2×2 satisfaisant $D_{11} = D_{22}$ et $D_{12} = -D_{21}$ par Cauchy-Riemann et qui est non-nulle en U. Par le théorème de la fonction inverse (Série 1 Exercise 3), la fonction inverse f en $w = f(z)$ a comme Jacobienne la matrice inverse $D^{-1} = (\det D)^{-1} \begin{pmatrix} D_{22} & -D_{21} \\ -D_{12} & D_{11} \end{pmatrix}$. On voit en particulier qu'elle satisfait également les équations de Cauchy-Riemann. □

<u>Exercice 6.</u>

(1) Montrer qu'il n'existe aucune fonction $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ t.q. $f(z^2) = z \ \forall z \in \mathbb{C} \setminus \{0\}$;

(2) Montrer qu'il n'existe aucune fonction continue $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ t.q. $f(z)^2 = z \ \forall z \in \mathbb{C} \setminus \{0\}$;

(3) Montrer qu'il n'existe pas de logarithme continu sur $\mathbb{C} \setminus \{0\}$.

Démonstration. (1) Il faudrait que $f(1^2) = 1$ et $f((-1)^2) = -1$: contradiction.

(2) Suppose such a function exists. Note that f maps the unit circle onto itself. Since f is continuous, there exists a continuous function $\phi : [0, 2\pi) \to [0, 2\pi)$ such that $f(e^{it}) = e^{i\phi(t)}$ for all $t \in [0, 2\pi)$. Furthermore, according to our assumptions $e^{it} = f(e^{it})^2 = e^{i2\phi(t)}$, which implies that $\phi(t) = t/2 + k\pi$ with $k \in \{0, 1\}$. Since $\phi(t) - t/2$ is continuous (difference of continuous functions) and takes values in $\{0, \pi\}$, $\phi(t) - t/2$ must be constant which gives us $\phi(t) = \phi(0) + t/2$. We have $\lim_{t \to 2\pi} \phi(t) = \phi(0) + \pi$ and by continuity

$f(1) = f(e^{2\pi i}) = e^{i\phi(0)}e^{i\pi} = -e^{i\phi(0)}$ and $f(1) = f(e^{i0}) = e^{i\phi(0)}e^{i0/2} = e^{i\phi(0)}$, which would imply $e^{i\phi(0)} = 0$ and this is not possible.

(3) Suppose there exists a continuous function $h : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ such that $e^{h(z)} = z$. Then the function $f(z) = e^{\frac{1}{2}h(z)}$ is continuous and satisfies $f(z)^2 = e^{h(z)} = z$ for all $z \in \mathbb{C} \setminus \{0\}$. This contradicts part (2) of the exercise.

$\square$

Exercice 7. Trouver le rayon de convergence des séries suivantes :

(1) $\sum_{n=0}^{\infty} z^{n!}$

(2) $\sum_{n=0}^{\infty} n^{(-1)^n} z^n$

Démonstration. (1) Le rayon de convergence est 1 car pour chaque $\rho \in [0, 1)$,

$$\sum_{n=0}^{\infty} \rho^{n!} \leq \sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho} < \infty$$

et pour chaque $\rho \geq 1$, chacun des termes $\rho^{n!}$ ne convergent pas vers 0 et il ne peut donc pas y avoir de convergence.

(2) On a $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} n^{\frac{(-1)^n}{n}} = 1$, et donc le rayon de convergence est 1.

$\square$

Exercice 8. Soit $f(z) = \sum_{k=0}^{\infty} a_k z^k$ une série convergente avec rayon de convergence $R > 0$. Soit $n \in \mathbb{N}$ et $j \in \{0, \ldots, n-1\}$. Montrer que pour $|z| < R$,

$$\sum_{k=0}^{\infty} a_{j+kn} z^{j+kn} = \frac{1}{n} \sum_{u=0}^{n-1} e^{-\frac{2\pi i}{n}uj} f(e^{\frac{2\pi i}{n}u} z) \ .$$

Démonstration. We have

$$\frac{1}{n} \sum_{u=0}^{n-1} e^{-\frac{2\pi i}{n}uj} f\left(e^{\frac{2\pi i}{n}u} z\right) = \frac{1}{n} \sum_{u=0}^{n-1} e^{-\frac{2\pi i}{n}uj} \sum_{m=0}^{\infty} a_m e^{\frac{2\pi i m}{n}u} z^m = \sum_{m=0}^{\infty} a_m z^m \left(\frac{1}{n} \sum_{u=0}^{n-1} e^{\frac{2\pi i}{n}u(m-j)} \right).$$

The sum $\sum_{u=0}^{n-1} e^{\frac{2\pi i}{n}u(m-j)}$ equals to n if $m-j$ is divisible by n, and to $\frac{1-e^{2\pi i(m-j)}}{1-e^{\frac{2\pi i}{n}(m-j)}} = 0$ otherwise since $m-j$ is an integer. Therefore the above expression equals $\sum_{k=0}^{\infty} a_{j+kn} z^{j+kn}$. $\square$